JOURNAL OF APPROXIMATION THEORY 19, 83–99 (1977)

Multioptimum of a Convex Functional

D. V. PAI

Department of Mathematics, Indian Institute of Technology, Bombay, Powai, Bombay-400076, India, and Department of Mathematics, Technological University, Eindhoven, The Netherlands

Communicated by R. C. Buck

Received November 1, 1975

1. INTRODUCTION

Let U, V be a pair of convex sets in a normed linear space X. The points $\bar{u} \in U$, $\bar{v} \in V$ are called proximal if

$$\|\bar{u} - \bar{v}\| = d(U, V) = \inf_{u \in U, v \in V} \|u - v\|.$$

It is easily observed that if the points $\bar{u} \in U$, $\bar{v} \in V$ are proximal, then they are mutually nearest to each other from the respective sets. However, the converse implication is generally not true, even for Chebyshev[1] sets U, V. In this connection it is convenient to restate here the following result from Pai [9]: "In order that for each pair U, V of convex sets in X, points $\bar{u} \in U$, $\bar{v} \in V$ that are mutually nearest to each other be proximal, it is necessary and sufficient that the space X be smooth." In the present paper this result is embedded in the answers to the following general questions pertaining to convex optimization in locally convex spaces.

QUESTION 1. Let f be a convex functional defined on a Hausdorff locally convex linear topological space X. Let U, V be a pair of convex sets in X. A pair $(\bar{u}, \bar{v}) \in U \times V$ is called a *multioptimum* for f if

$$f(\bar{u} - \bar{v}) = \inf_{v \in V} f(\bar{u} - v) = \inf_{u \in U} f(u - \bar{v}), \tag{1.1}$$

and it is called simply an *optimum* for f if $\bar{u} - \bar{v}$ is an *optimum* for f on $U - V$, i.e.,

$$f(\bar{u} - \bar{v}) = \inf_{u \in U, v \in V} f(u - v). \tag{1.2}$$

[1] See [12, p. 103] for the definition.

ISSN 0021-9045

Then we ask: Under what conditions is a *multioptimum* $(\bar{u}, \bar{v})$ an *optimum* for f? More generally, we are concerned with

QUESTION 2. Let X_i, $i = 1, 2,..., n$, be Hausdorff locally convex linear topological spaces and let $K_i \subset X_i$, $i = 1, 2,..., n$, be convex sets. Let F be a convex functional defined on $\prod_{i=1}^{n} X_i$. Denote by $F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}$, $i = 1, 2,..., n$, the convex functionals defined on X_i by

$$F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(x_i) = F(\bar{x}_1, \bar{x}_2,..., \bar{x}_{i-1}, x_i, \bar{x}_{i+1},..., \bar{x}_n). \tag{1.3}$$

We call $(\bar{x}_1, \bar{x}_2,..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ a *multioptimum* for F if

$$F(\bar{x}_1,..., \bar{x}_n) = \inf_{x \in K_i} F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(x_i), \qquad i = 1, 2,..., n, \tag{1.4}$$

and simply an *optimum* for F if

$$F(\bar{x}_1,..., \bar{x}_n) = \inf_{\substack{x_i \in K_i \\ i=1,2,...,n}} F(x_1, x_2,..., x_n). \tag{1.5}$$

Then we ask: Under what conditions is a *multioptimum* $(\bar{x}_1,..., \bar{x}_n)$ an *optimum* for F?

Question 2, of course, contains Question 1 as a special case upon taking $F(u, v) = f(u - v)$. However, as it turns out, for particular cases such as the case when f is a gauge function, necessary as well as sufficient conditions can be given in order that, for each pair U, V of convex sets, $(\bar{u}, \bar{v}) \in U \times V$ being a *multioptimum* for f imply that it is an *optimum* for f.

The main results pertaining to Question 2 are given in Section 2. In Section 3 we are concerned with Question 1 and also deal there with a special case when f is given as a certain seminorm. Section 4 deals with the important special cases when the convex sets K_i of Section 2 and the convex sets U, V of Section 3 are contained in subspaces of finite dimension. In Section 5 we discuss two applications: (1) multivariate constrained convex optimization, and (2) global simultaneous approximation.

We take the standard framework of convex analysis as adopted in [8] or [11] and recall here those notions that will frequently be used in the sequel. Let X, Y be complex linear spaces in duality, $\langle , \rangle$ denoting the duality relation. For topologies on X and Y, we take topologies compatible with the given duality $\langle , \rangle$. Equipped with these, X, Y become Hausdorff locally convex linear topological spaces. We say $f \in \operatorname{conv}(X)$ if $f: X \to \mathbb{R} \cup \{+\infty\}$ is proper, i.e., $f \not\equiv +\infty$ and it is convex. Let χ_K stand for the indicator function

$$\begin{aligned} \chi_K(x) &= 0, \qquad x \in K, \\ &= \infty, \qquad x \notin K, \end{aligned}$$

and let $\mathrm{dom}(f) = \{x \in X / f(x) < \infty\}$. Following Laurent [6, p. 335], we say that a function $f \in \mathrm{conv}(X)$ is d-continuous if f is continuous on int-dom(f). The subdifferential of f at $\bar{x}$ is $\partial f(\bar{x}) = \{y \in Y / f(x) \geqslant f(\bar{x}) + \mathrm{Re}\langle x - \bar{x}, y\rangle, \forall x \in X\}$. The following result of Moreau and Rockafellar (cf. Holmes [3, p. 25]) will frequently be employed. Let $f_1, f_2 \in \mathrm{conv}(X)$. Suppose there exists some point in $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$ at which one of the two functions is continuous. Then for each $\bar{x} \in X$ one has $\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x})$.

2. Characterization of Optimum and Multioptimum in Question 2

Let the linear spaces X_i and Y_i be in duality, $\langle , \rangle_i$ denoting the duality relation between them, $i = 1, 2, \ldots, n$. For the product spaces $\prod_{i=1}^n X_i$ and $\prod_{i=1}^n Y_i$ we take the following duality that corresponds to the given dualities between X_i and Y_i:

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = \langle x_1, y_1\rangle_1 + \langle x_2, y_2\rangle_2 + \cdots + \langle x_n, y_n\rangle_n .$$

THEOREM 2.1. *Let* $F \in \mathrm{conv}(\prod_{i=1}^n X_i)$ *and let* $K_i \subset X_i$ *be convex,* $i = 1, 2, \ldots, n$. *Assume that either*

(H_1) $\mathrm{dom}(F) \cap \prod_{i=1}^n \mathrm{int}(K_i) \neq \varnothing$

or

(H_1') *F is* d-*continuous and* int-dom$(F) \cap \prod_{i=1}^n K_i \neq \varnothing$

holds. Then $(\bar{x}_1, \ldots, \bar{x}_n) \in \prod_{i=1}^n K_i$ *is an optimum for F if and only if there exists* $(y_1, y_2, \ldots, y_n) \in \prod_{i=1}^n Y_i$ *such that*

(i) $(y_1, y_2, \ldots, y_n) \in \partial F(\bar{x}_1, \ldots, \bar{x}_n)$ *and*

(ii) $\mathrm{Re}\langle \bar{x}_i, y_i\rangle_i = \min_{x_i \in K_i} \mathrm{Re}\langle x_i, y_i\rangle_i$.

Proof. We observe that $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \in \prod_{i=1}^n K_i$ is an optimum for F

$$\text{iff } (0, 0, \ldots, 0) \in \partial(F + \chi_{\prod_{i=1}^n K_i})(\bar{x}_1, \ldots, \bar{x}_n),$$

$$\text{iff } \partial F(\bar{x}_1, \ldots, \bar{x}_n) \cap -\partial \chi_{\prod_{i=1}^n K_i}(\bar{x}_1, \ldots, \bar{x}_n) \neq \varnothing$$

(under hypothesis (H_1) or (H_1'). It suffices therefore to prove that

$$\partial \chi_{\prod_{i=1}^n K_i}(\bar{x}_1, \ldots, \bar{x}_n) = \prod_{i=1}^n \partial \chi_{K_i}(\bar{x}_i). \tag{2.1}$$

Indeed,

$$(y_1, \ldots, y_n) \in \partial\chi_{\prod_{i=1}^n K_i}(\bar{x}_1, \ldots, \bar{x}_n)$$

$$\text{iff} \max_{\substack{x_i \in K_i \\ i=1,2,\ldots,n}} \operatorname{Re}(\langle x_1, y_1\rangle_1 + \cdots + \langle x_n, y_n\rangle_n)$$

$$= \operatorname{Re}\langle \bar{x}_1, y_1\rangle_1 + \cdots + \operatorname{Re}\langle \bar{x}_n, y_n\rangle_n,$$

$$\text{iff} \max_{x_i \in K_i} \operatorname{Re}\langle x_i, y_i\rangle_i = \operatorname{Re}\langle \bar{x}_i, y_i\rangle_i, \qquad i = 1, 2, \ldots, n,$$

$$\text{iff } (y_1, \ldots, y_n) \in \prod_{i=1}^n \partial\chi_{K_i}(\bar{x}_i).$$

Theorem 2.1 is a slight extension of a theorem of Pšeničnii and Rockafellar for convex programs (see, e.g., [3, p. 30]).

THEOREM 2.2. *Suppose* $F \in \operatorname{conv}(\prod_{i=1}^n X_i)$ *and that it is finite and continuous at* $(\bar{x}_1, \ldots, \bar{x}_n) \in \prod_{i=1}^n K_i$. *Then in order that* $(\bar{x}_1, \ldots, \bar{x}_n)$ *being a multioptimum for F imply that it is an optimum for F, it is sufficient that the following equality hold for the subdifferentials*:

$$\partial F(\bar{x}_1, \ldots, \bar{x}_n) = \prod_{i=1}^n \partial F_{\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_n}(\bar{x}_i). \tag{2.2}$$

Proof. We first note the following easily established inclusion for the subdifferentials:

$$\partial F(\bar{x}_1, \ldots, \bar{x}_n) \subset \prod_{i=1}^n \partial F_{\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_n}(\bar{x}_i). \tag{2.3}$$

Assume now that equality (2.2) holds in the above inclusion. In view of (1.4) we have that $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is a multioptimum for F

$$\text{iff } 0 \in \partial(F_{\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_n} + \chi_{K_i})(\bar{x}_i)$$

$$= \partial F_{\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_n}(\bar{x}_i) + \partial\chi_{K_i}(\bar{x}_i), \qquad i = 1, 2, \ldots, n,$$

$$\text{iff } \prod_{i=1}^n \partial F_{\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_n}(\bar{x}_i) \cap \prod_{i=1}^n \partial\chi_{-K_i}(-\bar{x}_i) \neq \varnothing.$$

Employing (2.1) and (2.2), the last condition holds

$$\text{iff } \partial F(\bar{x}_1, \ldots, \bar{x}_n) \cap -\partial\chi_{\prod_{i=1}^n K_i}(\bar{x}_1, \ldots, \bar{x}_n) \neq \varnothing,$$

iff $(\bar{x}_1, \ldots, \bar{x}_n)$ is an optimum for F. ∎

Remarks. (1) Let $F \in \operatorname{conv}(\prod_{i=1}^n X_i)$ and let it be finite and continuous at $(\bar{x}_1, ..., \bar{x}_n)$. Then the equality (2.2) holds for the subdifferentials if and only if the following equality holds for the directional derivatives:

$$\begin{aligned} F'(\bar{x}_1, ..., \bar{x}_n ; x_1, ..., x_n) &= \sum_{i=1}^n F'_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(\bar{x}_i ; x_i) \\ &= \sum_{i=1}^n F'(\bar{x}_1, ..., \bar{x}_n ; 0, 0, ..., x_i, ..., 0). \end{aligned} \tag{2.4}$$

In fact,

$$\begin{aligned} F'(\bar{x}_1, ..., \bar{x}_n ; x_1, ..., x_n) &= \max_{(y_1,...,y_n)\in\partial F(\bar{x}_1,...,\bar{x}_n)} \left\{ \sum_{i=1}^n \operatorname{Re}\langle x_i , y_i\rangle \right\} \\ &\leqslant \sum_{i=1}^n \max_{y_i \in \partial F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(\bar{x}_i)} \operatorname{Re}\langle x_i , y_i\rangle \\ &= \sum_{i=1}^n F'_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(\bar{x}_i ; x_i). \end{aligned}$$

Hence, equality in the inclusion (2.3) for subdifferentials enforces equality in the above inequality. Conversely, suppose (2.4) holds and let

$$(y_1, y_2, ..., y_n) \in \prod_{i=1}^n \partial F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(\bar{x}_i).$$

Then

$$\begin{aligned} \sum_{i=1}^n \operatorname{Re}\langle x_i , y_i\rangle &\leqslant \sum_{i=1}^n F'_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(\bar{x}_i ; x_i) = F'(\bar{x}_1, ..., \bar{x}_n ; x_1, ..., x_n) \\ &\leqslant F(x_1 + \bar{x}_1, ..., x_n + \bar{x}_n) - F(\bar{x}_1, ..., \bar{x}_n), \\ &\qquad x_i \in X_i, \ i = 1, 2, ..., n. \end{aligned}$$

Thus $(y_1, y_2, ..., y_n) \in \partial F(\bar{x}_1, ..., \bar{x}_n)$ and (2.2) holds.

(2) Again, let $F \in \operatorname{conv}(\prod_{i=1}^n X_i)$ and let it be finite and continuous at $(\bar{x}_1, ..., \bar{x}_n)$. Then F is Gâteaux-differentiable at $(\bar{x}_1, ..., \bar{x}_n)$ if and only if $F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}$ is Gâteaux-differentiable at $\bar{x}_i$, $i = 1, 2, ..., n$. In this case (2.2) evidently holds for the subdifferentials.

(3) Apart from the differentiable case of the preceding remark, another simple case, wherein (2.2) holds for the subdifferentials, is the following: $F(x_1, ..., x_n) = f_1(x_1) + \cdots + f_n(x_n)$, where $f_i \in \operatorname{conv}(X_i)$, $i = 1, 2, ..., n$.

(4) Condition (2.2) is not necessary in order for a multioptimum to be an optimum. To illustrate this, let $X_i = X$ be a Banach space, $i = 1, 2,..., n$, and let $F(x_1,..., x_n) = \| \sum_{i=1}^n x_i \|$. Let $(\bar{x}_1,..., \bar{x}_n) = (0, 0,..., 0)$. Then for arbitrarily given convex sets $K_i \subset X_i$ such that $(0,...,0) \in \prod_{i=1}^n K_i$, $(0, 0,..., 0)$ is a multioptimum implies that it is an optimum for F. However, in this case it is easily verified that (2.2) does not hold.

THEOREM 2.3. *Let $F \in \operatorname{conv}(\prod_{i=1}^n X_i)$ and let it be finite and continuous at $(\bar{x}_1,..., \bar{x}_n)$. Furthermore, suppose that $(0,..., 0) \notin \partial F(\bar{x}_1,..., \bar{x}_n)$ and that the following holds*:

$$\Bigg(\underbrace{\prod_{i=1}^{n} \bigcup_{\lambda_i > 0}}_{(\lambda_1,..., \lambda_n) \neq (1,..., 1)} \lambda_i\, \partial F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(\bar{x}_i)\Bigg) \cap \partial F(\bar{x}_1,..., \bar{x}_n) = \varnothing . \tag{2.5}$$

Then in order that for arbitrarily given convex sets $K_i \subset X_i$ such that $\bar{x}_i \in K_i$, $i = 1, 2,..., n$, $(\bar{x}_1,..., \bar{x}_n)$ being a multioptimum for F imply that it is an optimum for F, it is necessary and sufficient that (2.2) *hold.*

Proof. The sufficiency part is already contained in Theorem 2.2. In order to prove the necessity part, suppose that (2.2) does not hold and let

$$(\bar{y}_1,..., \bar{y}_n) \in \Bigg(\prod_{i=1}^{n} \partial F_{\bar{x}_1,...,\bar{x}_{i-1},\bar{x}_{i+1},...,\bar{x}_n}(\bar{x}_i)\Bigg) \backslash \partial F(\bar{x}_1,..., \bar{x}_n). \tag{2.6}$$

Define the convex sets K_i as follows:

$$K_i = \{x_i \in X_i / \operatorname{Re}\langle x_i - \bar{x}_i, \bar{y}_i\rangle \geqslant 0\}, \qquad i = 1, 2,..., n.$$

Then by Theorem 2.1 one has that $(\bar{x}_1,..., \bar{x}_n)$ is a multioptimum for F on $\prod K_i$. To complete the proof of the theorem, we assert that $(\bar{x}_1,..., \bar{x}_n)$ is not an optimum for F on $\prod K_i$.

Assume the contrary. Then using Theorem 2.1 once more there exists an element

$$(\tilde{y}_1,..., \tilde{y}_n) \in \partial F(\bar{x}_1,..., \bar{x}_n) \cap \prod_{i=1}^{n} \partial\chi_{-K_i}(-\bar{x}_i).$$

Now let

$$H_i = \{x_i \in X_i / \operatorname{Re}\langle x_i - \bar{x}_i, \tilde{y}_i\rangle \geqslant 0\}, \qquad i = 1, 2,..., n.$$

Since $\tilde{y}_i \in \partial\chi_{-K_i}(-\bar{x}_i)$, one has $K_i \subset H_i$, $i = 1,..., n$. This inclusion of the half-spaces in X_i entails that $\tilde{y}_i = \lambda_i \bar{y}_i$, $\lambda_i > 0$, $i = 1, 2,..., n$, where not all the λ_i's are equal to 1 on account of (2.6). This contradicts (2.5) and establishes the theorem. ∎

3. Characterization of Optimum and Multioptimum in Question 1

LEMMA 3.1. *Let* $f \in \operatorname{conv}(X)$ *and define* $F \in \operatorname{conv}(X \times X)$ *by* $F(x_1, x_2) = f(x_1 - x_2)(x_1, x_2 \in X)$. *Given* $\bar{x}_1, \bar{x}_2 \in X$, *one has*

$$\partial F_{\bar{x}_2}(\bar{x}_1) = \partial f(\bar{x}_1 - \bar{x}_2), \qquad \partial F_{\bar{x}_1}(\bar{x}_2) = -\partial f(\bar{x}_1 - \bar{x}_2) \tag{3.1}$$

and

$$\partial F(\bar{x}_1, \bar{x}_2) = \{(y_1, y_2)/y_2 = -y_1, y_1 \in \partial f(\bar{x}_1 - \bar{x}_2)\}. \tag{3.2}$$

Proof. Relations (3.1) and the inclusion

$$\{(y_1, y_2)/y_2 = -y_1, y_1 \in \partial f(\bar{x}_1 - \bar{x}_2)\} \subset \partial F(\bar{x}_1, \bar{x}_2)$$

are obvious.

In order to reverse this inclusion, suppose $(y_1, y_2) \in \partial F(\bar{x}_1, \bar{x}_2)$. Then in view of (2.3) and (3.1) one has $(y_1, y_2) \in \partial f(\bar{x}_1 - \bar{x}_2) \times -\partial f(\bar{x}_1 - \bar{x}_2)$. Taking into account the definition of subdifferential the inequality

$$f(x_1 - x_2) \geqslant f(\bar{x}_1 - \bar{x}_2) + \operatorname{Re}\langle x_1 - \bar{x}_1, y_1\rangle + \operatorname{Re}\langle x_2 - \bar{x}_2, y_2\rangle$$

holds for all $x_1, x_2 \in X$. Hence, in particular it holds for $x_1, x_2 \in X$ satisfying $x_1 - \bar{x}_1 = x_2 - \bar{x}_2$. Thus for all $x \in X$ we have $\operatorname{Re}\langle x, y_1 + y_2\rangle \leqslant 0$, which yields $y_1 + y_2 = 0$.

Employing Lemma 3.1 and Theorem 2.1, one immediately obtains

THEOREM 3.2. *Let* $f \in \operatorname{conv}(X)$ *and let* U, V *be convex sets in* X. *Assume that either*

(H_2) $\operatorname{dom}(f) \cap \operatorname{int}(U - V) \neq \varnothing$

or

(H_2') f *is* d-*continuous and* $\operatorname{int}\operatorname{dom}(f) \cap (U - V) \neq \varnothing$

holds. Then $(\bar{u}, \bar{v}) \in U \times V$ *is an optimum for* f *if and only if there exists an element* $y \in Y$ *such that*

(i) $y \in \partial f(\bar{u} - \bar{v})$,

(ii) $\operatorname{Re}\langle \bar{u}, y\rangle = \inf_{u \in U} \operatorname{Re}\langle u, y\rangle$,

(iii) $\operatorname{Re}\langle \bar{v}, y\rangle = \sup_{v \in V} \operatorname{Re}\langle v, y\rangle$.

The next theorem furnishes an answer to Question 1 as a particular case of Theorem 2.2.

THEOREM 3.3. *Let* $f \in \operatorname{conv}(X)$ *and let it be finite and continuous at* $\bar{u} - \bar{v}$. *Then in order that* $(\bar{u}, \bar{v}) \in U \times V$ *being a multioptimum for* f *imply that it is an optimum for* f, *it is sufficient that* f *be Gâteaux-differentiable at* $\bar{u} - \bar{v}$.

Proof. Due to the assumption that f is finite and continuous at $\bar{u} - \bar{v}$, we note that $\partial f(\bar{u} - \bar{v})$ consists of a single element if and only if f is Gâteaux-differentiable at $\bar{u} - \bar{v}$. Moreover, in view of Lemma 3.1 and the above observation it follows that if we take $F(u, v) = f(u - v)$, then the equality $\partial F(\bar{u}, \bar{v}) = \partial F_{\bar{v}}(\bar{u}) \times \partial F_{\bar{u}}(\bar{v})$ holds for the subdifferentials if and only if f is Gâteaux-differentiable at $\bar{u} - \bar{v}$. The proof is completed by applying Theorem 2.2. ∎

Remark. Theorem 3.3 remains valid if instead of assuming that f is finite and contnuous at $\bar{u} - \bar{v}$, we make any one of the following weaker hypotheses:

(H_3) $(\bar{u} - \text{dom}(f)) \cap \text{int } V \neq \varnothing$ and $(\bar{v} + \text{dom}(f)) \cap \text{int } U \neq \varnothing$;

(H_3') f is d-continuous and $\text{int}(\bar{u} - \text{dom}(f)) \cap V \neq \varnothing$, $\text{int}(\bar{v} + \text{dom}(f)) \cap U \neq \varnothing$.

The assertion follows easily by taking into account the first observation in the proof of Theorem 3.3 and then employing Theorem 3.2.

COROLLARY 3.4. *Let $f \in \text{conv}(X)$ and let it be finite and continuous at $\bar{x} \in X$. Further suppose that $0 \notin \partial f(\bar{x})$ and that $\{\bigcup_{\lambda>0,\lambda\neq 1} \lambda\partial f(\bar{x})\} \cap \partial f(\bar{x}) = \varnothing$. Then in order that, for arbitrary given convex sets U, V such that $\bar{x} \in U$ and $0 \in V$, $(\bar{x}, 0)$ being a multioptimum for f imply that it is an optimum for f, it is necessary and sufficient that f be Gâteaux-differentiable at $\bar{x}$.*

Proof. This follows immediately from Theorem 2.3 and Lemma 3.1. ∎

Corollary 3.4 can be strengthened in the case where f is a gauge function on X, i.e., a real-valued function on X satisfying $f(x_1 + x_2) \leqslant f(x_1) + f(x_2)$ for all $x_1, x_2 \in X$ and $f(\lambda x) = \lambda f(x)$ for all $x \in X$ and $\lambda \geqslant 0$. In this case the subdifferential of f at $\bar{x}$ is given by $\partial f(\bar{x}) = \{y \in \partial f(\theta) / f(\bar{x}) = \text{Re}\langle\bar{x}, y\rangle\}$, where θ denotes the zero vector of X. One thus obtains

THEOREM 3.5. *Let f be a continuous gauge function defined on X. Then in order that for arbitrarily given convex sets U, V in X and points $\bar{u} \in U$, $\bar{v} \in V$ such that $f(\bar{u} - \bar{v}) \neq 0$, $(\bar{u}, \bar{v})$ being a multioptimum for f imply that it is an optimum for f, it is necessary and sufficient that f be Gâteaux-differentiable at each point $x \in X$, where $f(x) \neq 0$.*

Proof. The sufficiency part is already contained in Theorem 3.3. To prove the necessity part, suppose there exists a point $\bar{x} \in X$ such that $f(\bar{x}) \neq 0$ and such that f is not Gâteaux-differentiable at $\bar{x}$. Then there exist $y_1, y_2 \in \partial f(\theta)$, $y_1 \neq y_2$, such that $\text{Re}\langle\bar{x}, y_1\rangle = \text{Re}\langle\bar{x}, y_2\rangle = f(\bar{x})$. Let us assume first that $f(\bar{x}) > 0$. Now select $\tilde{x} \in X$ such that $0 < \text{Re}\langle\tilde{x}, y_1\rangle <$

$\mathrm{Re}\langle\tilde{x}, y_2\rangle$ and let $U = \{x/\mathrm{Re}\langle x, y_2\rangle \geqslant f(\bar{x})\}$, $V = \{x/\mathrm{Re}\langle x, y_1\rangle = 0\}$. Then $\bar{x} \in U$, $0 \in V$ and $(\bar{x}, 0)$ is a multioptimum for f, but it is not an optimum for f.

In fact, let $\hat{x} = \tilde{x}f(\bar{x})/\mathrm{Re}\langle\tilde{x}, y_2\rangle$. Then $\mathrm{Re}\langle\hat{x}, y_1\rangle < f(\bar{x}) = \mathrm{Re}\langle\hat{x}, y_2\rangle$. This gives $\hat{x} \in U$, $\hat{x} - \mathrm{Re}\langle\hat{x}, y_1\rangle(\bar{x}/f(\bar{x})) \in V$ and

$$f\left(\hat{x} - \left(\hat{x} - \mathrm{Re}\langle\hat{x}, y_1\rangle \frac{\bar{x}}{f(\bar{x})}\right)\right) = \mathrm{Re}\langle\hat{x}, y_1\rangle < f(\bar{x}).$$

In case $f(\bar{x}) < 0$ we select $\tilde{x} \in X$ such that $0 < \mathrm{Re}\langle\tilde{x}, y_2\rangle < \mathrm{Re}\langle\tilde{x}, y_1\rangle$ and proceed exactly as before. ∎

In the last part of this section we consider the particular case when f is a seminorm defined as follows. Let B be a balanced and equicontinuous subset of Y and let

$$f(x) = \sup_{y \in B} \mathrm{Re}\langle x, y\rangle, \qquad x \in X.$$

We note that the set $K = \overline{\mathrm{co}}(B)$ is a balanced convex and $\sigma(Y, X)$-compact subset of Y and hence we have $f(x) = \max_{y\in K} \mathrm{Re}\langle x, y\rangle$ and

$$\partial f(\bar{x}) = \{y \in K/f(\bar{x}) = \mathrm{Re}\langle\bar{x}, y\rangle\}.$$

In the case of a real locally convex space X, the above seminorm f has been employed by Laurent [6, p. 426].

COROLLARY 3.6. *Suppose that K is not contained in any closed hyperplane,* core[2] $(K) \neq \varnothing$ *and that K is strictly convex, i.e., $x_1, x_2 \in K$, $0 < \lambda < 1$, imply* $(1 - \lambda) x_1 + \lambda x_2 \in \mathrm{core}(K)$. *Then for arbitrarily given convex sets U, V in X and points $\bar{u} \in U$, $\bar{v} \in V$, $(\bar{u}, \bar{v})$ is a multioptimum for f implies that it is an optimum for f.*

Proof. In view of Theorem 3.5 it suffices to prove that for each $x \in X$, $x \neq \theta$, $\partial f(x)$ consists of a single element. We note that for $\theta \neq x \in X$ the set $\partial f(x)$ is a nonempty $\sigma(Y, X)$-compact and proper extremal subset of K. Hence, if $\partial(x)$ contained more than one point, say the points y_1 and y_2 with $y_1 \neq y_2$, then $(1 - \lambda) y_1 + \lambda y_2 \notin \mathrm{core}(K)$, $0 < \lambda < 1$. This contradicts the strict convextity of K and establishes the corollary. ∎

4. Characterization of Optimum for the Case of Finite Dimensional Convex Sets

Here we consider the important special cases when the convex sets K_i of Section 1 and the convex sets U, V of Section 3 are contained in subspaces

[2] Recall that $\mathrm{Core}(K) = \{k \in K/\forall_{k'\in Y}, \exists_{\epsilon>0} \ni \forall_{\lambda\in[-\epsilon,+\epsilon]}, k + \lambda k' \in K\}$.

of finite dimension. For this purpose we adopt essentially the approach as given in [6, Theorem 8.3.3, p. 438]. However, with a suitable modification of arguments, it is shown that Theorem 8.3.3 of [6] holds for a convex set in place of a linear subspace and thereby extends to cover the cases of optima dealt with in Section 2 and Section 3.

Let $K \subset X$ be convex and let $\bar{x} \in K$. By

$$C(K; \bar{x}) = \overline{\bigcup_{\lambda>0} \lambda(K - \bar{x})}$$

we denote the support cone of K at $\bar{x}$. Let $L(K; \bar{x}) = C(K; \bar{x}) \cap -C(K; \bar{x})$ stand for the largest subspace contained in the support cone $C(K; \bar{x})$. By the facet of $\bar{x}$ in K we will mean the set $H(K; \bar{x}) = (\bar{x} + L(K; \bar{x})) \cap K$. Note that $H(K; \bar{x})$ is the smallest extremal subset of K containing $\bar{x}$. Hence, $\bar{x}$ is and extreme point of K if and only if $H(K; \bar{x}) = \{\bar{x}\}$ (or equivalently $L(K; \bar{x}) = \{\theta\}$).

LEMMA 4.1. *Let X be of finite dimension n. Suppose $f \in \operatorname{conv}(X)$ and let it be finite and continuous at $\bar{x} \in X$. Suppose $y \in \partial f(\bar{x})$. Then there exist m elements $y_i \in \operatorname{Ext}^3 (\partial f(\bar{x}))$, $i = 1, 2, \ldots, m$, and m numbers $\lambda_i > 0$, $\sum_{i=1}^{m} \lambda_i = 1$, such that $y = \sum_{i=1}^{m} \lambda_i y_i$, with $1 \leqslant m \leqslant n + 1$ (for real scalars) or $1 \leqslant m \leqslant 2n + 1$ (for complex scalars).*

The lemma is well known (cf. [6, p. 436]). It is an immediate consequence of the Krein–Milman Theorem, a theorem of Carathéodory and the fact that $\partial f(\bar{x})$ is a nonempty $\sigma(Y, X)$-compact convex set.

THEOREM 4.2. *Let $F \in \operatorname{conv}(\prod_{i=1}^{n} X_i)$ and let it be finite and continuous at $(\bar{x}_1, \ldots, \bar{x}_n) \in \prod_{i=1}^{n} K_i$, where $K_i \subset X_i$ are convex sets such that $\dim[K_i]^4 = m_i$, $i = 1, \ldots, n$. Then $(\bar{x}_1, \ldots, \bar{x}_n)$ is an optimum for F if and only if there exist s elements $(y_1^{(j)}, \ldots, y_n^{(j)}) \in \operatorname{Ext} \partial F(\bar{x}_1, \ldots, \bar{x}_n)$, $1 \leqslant j \leqslant s$, and s numbers $\lambda_j > 0$, with $\sum_{j=1}^{s} \lambda_j = 1$ such that*

(i) *$1 \leqslant s \leqslant \sum_{i=1}^{n} m_i + 1$ (real scalars) or $1 \leqslant s \leqslant 2 \sum_{i=1}^{n} m_i + 1$ complex scalars),*

(ii) *$\operatorname{Re} \sum_{j=1}^{s} \lambda_j \langle \bar{x}_i - x_i, y_i^{(j)} \rangle_i \leqslant 0$ $(x_i \in K_i)$, $i = 1, 2, \ldots, n$.*

Proof. The sufficiency part of the theorem is trivial. To prove the necessity part, let $(\bar{x}_1, \ldots, \bar{x}_n) \in \prod_{i=1}^{n} K_i$ be an optimum for F. Then by Theorem 2.1 one has

$$\partial F(\bar{x}_1, \ldots, \bar{x}_n) \cap \prod_{i=1}^{n} -\partial \chi_{K_i}(\bar{x}_i) \neq \varnothing. \tag{4.1}$$

[3] As usual, $\operatorname{Ext}(A)$ denotes the set of extreme points of A.

[4] We employ the notation $[K_i]$ for the span of K_i.

We note that with the $\sigma(Y_i, X_i)$ topology on Y_i, $i = 1, 2,..., n$, the topology of the product for $\prod_{i=1}^n Y_i$ coincides with the $\sigma(\prod_{i=1}^n Y_i, \prod_{i=1}^n X_i)$ topology with respect to the corresponding duality (cf. [5, Theorem 17.14, p. 160]). Thus the set $\prod_{i=1}^n - \partial\chi_{K_i}(\bar{x}_i)$ is $\sigma(\prod Y_i, \prod X_i)$-closed and this entails that the set on the left-hand side of (4.1) is $\sigma(\prod Y_i, \prod X_i)$-compact. By the Krein–Milman theorem we have $\mathrm{Ext}(\partial F(\bar{x}_1,..., \bar{x}_n) \cap \prod_{i=1}^n - \partial\chi_{K_i}(\bar{x}_i)) \neq \varnothing$. Now let

$$(\bar{y}_1,..., \bar{y}_n) \in \mathrm{Ext}\left(\partial F(\bar{x}_1,..., \bar{x}_n) \cap \prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i)\right).$$

Then we have

$$\begin{aligned}\{(\theta,..., \theta)\} &= L\left(\partial F(\bar{x}_1,..., \bar{x}_n) \cap \prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_n)\right) \qquad (4.2)\\ &= L(\partial F(\bar{x}_1,..., \bar{x}_n); (\bar{y}_1,..., \bar{y}_n)) \cap L\left(\prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_n)\right)\end{aligned}$$

(cf. [6, Proposition 8.3.2]).

On the other hand,

$$\left(\prod_{i=1}^n [K_i]\right)^\perp \subset C\left(\prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_n)\right),$$

where $A^\perp$ stands for the annihilator subspace. For if we assume $(y_1,..., y_n) \in (\prod_{i=1}^n [K_i])^\perp$, then clearly

$$(y_1 + \bar{y}_1,..., y_n + \bar{y}_n) \in \prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i).$$

Hence, we get that

$$\left(\prod_{i=1}^n [K_i]\right)^\perp \subset L\left(\prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_2)\right).$$

Since

$$L\left(\prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_n)\right)$$

is the largest subspace contained in the cone

$$C\left(\prod_{i=1}^n -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1,..., \bar{y}_n)\right).$$

As a consequence, one has that

$$\operatorname{codim} L\left(\prod_{i=1}^{n} -\partial\chi_{K_i}(\bar{x}_i); (\bar{y}_1, ..., \bar{y}_n)\right) \leqslant \dim\left(\prod_{i=1}^{n} [K_i]\right) = \sum_{i=1}^{n} m_i . \quad (4.3)$$

From (4.2) and (4.3) we conclude that $L(\partial F(\bar{x}_1, ..., \bar{x}_n); (\bar{y}_1, ..., \bar{y}_n))$ is a subspace of dimension at most equal to $\sum_{i=1}^{n} m_i$. The remaining argument is exactly the same as that given in [6, Theorem 8.3.3]. In fact, $H(\partial F(\bar{x}_1, ..., \bar{x}_n); (\bar{y}_1, ..., \bar{y}_n))$ is a $\sigma(\prod Y_i, \prod X_i)$-compact convex set contained in a linear variety of dimension at most equal to $\sum_{i=1}^{n} m_i$ ($2\sum_{i=1}^{n} m_i$ for the complex case). Hence, to conclude the proof it only remains to apply Lemma 4.1 and to employ the fact that $H(\partial F(\bar{x}_1, ..., \bar{x}_n); (\bar{y}_1, ..., \bar{y}_n))$ is an extremal subset of $\partial F(\bar{x}_1, ..., \bar{x}_n)$. ∎

COROLLARY 4.3. *Let $f \in \operatorname{conv}(X)$ and let it be finite and continuous at $\bar{u} - \bar{v}$. Suppose $\bar{u} \in U$ and $\bar{v} \in V$, where U, V are convex sets such that* $\dim[U] = m$ *and* $\dim[V] = n$. *Then $(\bar{u}, \bar{v})$ is an optimum for f if and only if there exist k elements $y_i \in \operatorname{Ext} \partial f(\bar{u} - \bar{v})$ and k numbers $\lambda_i > 0$ with $\sum_{i=1}^{k} \lambda_i = 1$ such that*

(i) $1 \leqslant k \leqslant m + n + 1$, *(real scalars)*, $1 \leqslant k \leqslant 2m + 2n + 1$, *(complex scalars)*,

(ii) $\operatorname{Re} \sum_{i=1}^{k} \lambda_i \langle \bar{u} - u, y_i \rangle \leqslant 0 \ (u \in U)$,

(iii) $\operatorname{Re} \sum_{i=1}^{k} \lambda_i \langle \bar{v} - v, y_i \rangle \geqslant 0 \ (v \in V)$.

Proof. This follows immediately upon applying Lemma 3.1 and Theorem 4.2, ∎

Theorem 4.2 can be generalized so as to be valid under the slightly weaker hypothesis (H_1'). For this purpose we again adopt basically the same approach as that given in [6, Theorem 8.3.6].

We recall that an extremal ray D of a set $A \subset X$ is a closed semiline contained in A, which is also an extremal subset of A. Extreme directions of A are elements d such that A contains an extremal ray of the type $D = \{x/x = x_0 + \lambda d, \lambda \geqslant 0\}$.

THEOREM 4.4. *Let $F \in \operatorname{conv}(\prod_{i=1}^{n} X_i)$ and let it satisfy* (H_1'). *Let $K_i \subset X_i$ be convex sets such that* $\dim[K_i] = m_i$, $i = 1, ..., n$. *Then $(\bar{x}_1, ..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ is an optimum for F if and only if there exist s elements $(y_1^{(j)}, ..., y_n^{(j)}) \in \operatorname{Ext} \partial F(\bar{x}_1, ..., \bar{x}_n)$, $1 \leqslant j \leqslant s$ and t elements $(d_1^{(j)}, ..., d_n^{(j)})$, $1 \leqslant j \leqslant t$, $t \geqslant 0$, that are extreme directions of $\partial F(\bar{x}_1, ..., \bar{x}_n)$, with $1 \leqslant s + t \leqslant \sum_{i=1}^{n} m_i + 1$*

($2\sum_{i=1}^{n} m_i + 1$ *for complex scalars*) *and positive numbers* $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t$, $\sum_{j=1}^{s} \lambda_j = 1$, *such that*

$$\operatorname{Re}\left\{\sum_{j=1}^{s} \lambda_j \langle \bar{x}_i - x_i, y_j^{(i)} \rangle_i + \sum_{j=1}^{t} \mu_j \langle \bar{x}_i - x_i, d_i^{(j)} \rangle_i \right\} \leqslant 0$$

$$(x_i \in K_i), \quad i = 1, 2, \ldots, n.$$

Proof. The sufficiency part of the theorem clearly follows from Theorem 2.1. To prove the necessity part, let $(\bar{x}_1, \ldots, \bar{x}_n) \in \prod_{i=1}^{n} K_i$ be an optimum for F. Since F is d-continuous the set $\partial F(\bar{x}_1, \ldots, \bar{x}_n)$ is a $\sigma(\prod_{i=1}^{n} Y_i, \prod_{i=1}^{n} X_i)$-closed convex locally compact set not containing a line (cf. [4]). Employing Theorem 2.1 and Theorem 8.3.6(i) of Laurent [6, p. 441], there exists an element

$$(\bar{y}_1, \ldots, \bar{y}_n) \in \operatorname{Ext}\left(\partial F(\bar{x}_1, \ldots, \bar{x}_n) \cap \prod_{i=1}^{n} -\partial \chi_{K_i}(\bar{x}_i)\right).$$

Proceeding exactly as in the proof of Theorem 4.2, we obtain that $H(\partial F(\bar{x}_1, \ldots, \bar{x}_n); (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n))$ is a $\sigma(\prod_{i=1}^{n} Y_i, \prod_{i=1}^{n} X_i)$-closed convex set not containing a line and such that it is contained in a linear variety of dimension equal to $\sum_{i=1}^{n} m_i$ ($2\sum_{i=1}^{n} m_i$ for the complex case). The remaining argument is the same as that given in [6, Theorem 8.3.7]. ∎

COROLLARY 4.5. *Let* $f \in \operatorname{conv}(X)$ *and let it satisfy* (H_2'). *Let* U, V *be convex sets such that* $\dim[U] = m$, $\dim[V] = n$, *and let* $\bar{u} \in U$, $\bar{v} \in V$. *Then* $(\bar{u}, \bar{v})$ *is an optimum for* f *if and only if there exist* k *elements* y_i, $i = 1, \ldots, k$, $k \geqslant 1$, *that are extreme points of* $\partial f(\bar{u} - \bar{v})$ *and* s *elements* d_i, $i = 1, \ldots, s$, $s \geqslant 0$, *that are extreme directions of* $\partial f(\bar{u} - \bar{v})$ *with* $1 \leqslant k + s \leqslant m + n + 1$ ($2m + 2n + 1$ *for the complex case*) *and positive numbers* $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_s$, *such that*

(i) $\operatorname{Re}\{\sum_{i=1}^{k} \lambda_i \langle \bar{u} - u, y_i \rangle + \sum_{i=1}^{s} \mu_i \langle \bar{u} - u, d_i \rangle\} \leqslant 0 \ (u \in U)$,

(ii) $\operatorname{Re}\{\sum_{i=1}^{k} \lambda_i \langle \bar{v} - v, y_i \rangle + \sum_{i=1}^{s} \mu_i \langle \bar{v} - v, d_i \rangle\} \geqslant 0 \ (v \in V)$.

5. APPLICATIONS

Here we consider two specific examples, wherein the results of the previous sections are applicable.

a. *Multivariate Constrained Convex Optimization*

Let $F \in \operatorname{conv}(\prod_{i=1}^{n} X_i)$ and let it be finite everywhere. Let $f_i^{(j)} \in \operatorname{conv}(X_i)$ and let it be continuous on X_i, $j = 1, 2, \ldots, m_i$, $i = 1, 2, \ldots, n$. Furthermore, let the convex sets $K_i \subset X_i$ be defined as follows:

$$K_i = K_i^{(1)} \cap K_i^{(2)} \cap \cdots \cap K_i^{(m_i)}, \qquad i = 1, 2, \ldots, n,$$

where

$$K_i^{(j)} = \{x_i \in X_i / f_i^{(j)}(x_i) \leqslant 0\}, \qquad j = 1, 2, \ldots, m_i \text{ and } i = 1, 2, \ldots, n,$$

In addition, we make the following regularity hypothesis on the functions $f_i^{(j)}$:

$$(\mathrm{R}_1) \qquad \bigcap_{j=1}^{m_i} \overset{\circ}{K}_i^{(j)} \neq \varnothing, \qquad i = 1, 2, \ldots, n.$$

Then Theorem 2.1 takes the following particular form.

THEOREM 5.1. (Kuhn–Tucker-type characterization) *If the hypothesis* (R_1) *is fulfilled, then* $(\bar{x}_1, \ldots, \bar{x}_n) \in \prod_{i=1}^{n} K_i$ *is an optimum for* F *if and only if there exist elements* $y_i^{(j)} \in \partial f_i^{(j)}(x_i)$ *and numbers* $\lambda_i^{(j)} \leqslant 0$, $j = 1, 2, \ldots, m_i$, $i = 1, 2, \ldots, n$, *such that* $\lambda_i^{(j)} f_i^{(j)}(\bar{x}_i) = 0, j = 1, \ldots, m_i$, $i = 1, 2, \ldots, n$, *and*

$$\left(\sum_{j=1}^{m_1} \lambda_1^{(j)} y_1^{(j)}, \ldots, \sum_{j=1}^{m_n} \lambda_n^{(j)} y_n^{(j)}\right) \in \partial F(\bar{x}_1, \ldots, \bar{x}_n).$$

Proof. We have $\chi_{K_i} = \sum_{j=1}^{m_i} \chi_{K_i^{(j)}}$ and in view of the hypothesis (R_1) one has

$$\partial \chi_{K_i}(\bar{x}_i) = \sum_{j=1}^{m_i} \partial \chi_{K_i^{(j)}}(\bar{x}_i).$$

Furthermore, we note that in this case the subdifferential $\partial \chi_{K_i^{(j)}}(\bar{x}_i)$ has the following explicit expression:

$$\begin{aligned} \partial \chi_{K_i^{(j)}}(\bar{x}_i) &= \varnothing, && \text{if } f_i^{(j)}(\bar{x}_i) > 0, \\ &= \{\theta\}, && \text{if } f_i^{(j)}(\bar{x}_i) < 0, \\ &= -C(\theta, \partial f_i^{(j)}(\bar{x}_i)), && \text{if } f_i^{(j)}(\bar{x}_i) = 0, \end{aligned}$$

(cf. [3, p. 32].

To complete the proof it suffices to observe that by Theorem 2.1, $(\bar{x}_1, ..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ is an optimum for F if and only if

$$\partial F(\bar{x}_1, ..., \bar{x}_n) \cap \prod_{i=1}^{n} -\partial \chi_{K_i}(\bar{x}_i) \neq \varnothing. \quad \blacksquare$$

COROLLARY 5.2. *Let $f \in \operatorname{conv}(X)$ and let it be finite everywhere. Let $g_i \in \operatorname{conv}(X)$ and let it be continuous, $i = 1, 2, ..., l$. Let $h_j \in \operatorname{conv}(X)$ and let it be continuous, $j = 1, 2, ..., m$. Let the convex sets U, V be defined by $U = \bigcap_{i=1}^{l} U_i$, $V = \bigcap_{j=1}^{m} V_j$. Furthermore, suppose that the following regularity hypothesis is satisfied:*

$$(\mathrm{R}_2) \qquad \bigcap_{i=1}^{l} \mathring{U}_i \neq \varnothing \quad \text{and} \quad \bigcap_{j=1}^{m} \mathring{V}_j \neq \varnothing.$$

Then $(\bar{u}, \bar{v})$ is an optimum for f if and only if there exist

$$y_i \in \partial g_i(\bar{u} - \bar{v}), \qquad \lambda_i \leqslant 0, \quad i = 1, 2, ..., l,$$

and

$$y_j' \in \partial h_j(\bar{u} - \bar{v}), \qquad \lambda_j' \geqslant 0, \quad j = 1, 2, ..., m,$$

such that

$$\lambda_i g_i(\bar{u} - \bar{v}) = 0, \qquad i = 1, 2, ..., l, \quad \lambda_j' h_j(\bar{u} - \bar{v}) = 0, \quad j = 1, 2, ..., m,$$

and

$$\sum_{i=1}^{l} \lambda_i y_i = \sum_{j=1}^{m} \lambda_j' y_j' \in \partial f(\bar{u} - \bar{v}).$$

Proof. This follows immediately from Theorem 5.1 upon applying Lemma 3.1. $\blacksquare$

b. *Global Simultaneous Approximation*

Let X be a normed linear space and let $K_i \subset X$, $i = 1, 2, ..., n$, be convex sets. For $1 \leqslant p \leqslant \infty$ we consider the following optimization problems:

(Pb_p) Minimize $\{\|x_1 - x_2\|^p + \|x_1 - x_3\|^p + \cdots + \|x_1 - x_n\|^p\}^{1/p}$ for $x_i \in K_i$, $i = 1, 2, ..., n$, where $1 \leqslant p < \infty$.

(Pb_∞) Minimize $\{\max(\|x_1 - x_2\|, \|x_1 - x_3\|, ..., \|x_1 - x_n\|)\}$ for $x_i \in K_i$, $i = 1, 2, ..., n$.

For the case when $n = 2$ these problems coincide with the problem of determining proximal points of convex sets which has been dealt with in [10]. On the other hand, when each one of the sets $K_2, ..., K_n$ is reduced to a singleton set these problems coincide with the so-called l_p-problems of simultaneous approximation. In case $X = \mathscr{C}[a, b]$, the space of continuous functions with the uniform norm, and K_1 is taken to be a unisolvent family of degree n, the l_∞-problem of simultaneous approximation has been con-considered in [2]. A more general problem of global approximation of a compact set has been treated in [7]. Here we particularize Theorem 2.1 so as to obtain a characterization of solutions to the above problems.

THEOREM 5.3. *Let $p' = p/(p-1)$ if $1 < p < \infty$, $p' = \infty$ if $p = 1$ and $p' = 1$ if $p = \infty$. Then in order that $(\bar{x}_1, \ldots, \bar{x}_n) \in \prod_{i=1}^n K_i$ be a solution to the problem* (Pb$_p$), $1 \leqslant p \leqslant \infty$, *it is necessary and sufficient that there exist $y_i \in S(X^*)$, $S(X^*)$ being the unit sphere of X^*, $i = 1, 2, \ldots, n$, such that*

(i) $\sum_{i=1}^n y_i = 0$,

(ii) $(\sum_{i=2}^n \| y_i \|^{p'})^{1/p'} = 1$ (for $p' = \infty$, $\max_{2 \leqslant i \leqslant n} \| y_i \| = 1$),

(iii) $\operatorname{Re}\langle \bar{x}_i - x_i, y_i \rangle \leqslant 0$ $(x_i \in K_i)$, $i = 1, 2, \ldots, n$,

(iv) $\operatorname{Re} \sum_{i=1}^n \langle \bar{x}_i, y_i \rangle = (\sum_{i=2}^n \| \bar{x}_1 - \bar{x}_i \|^p)^{1/p}$ (for $p = \infty$, $\max_{2 \leqslant i \leqslant n} \| \bar{x}_1 - \bar{x}_i \|$).

Proof. We set

$$F(x_1, \ldots, x_n) = \left(\sum_{i=2}^n \| x_1 - x_i \|^p \right)^{1/p}, \qquad 1 \leqslant p < \infty,$$
$$= \max_{2 \leqslant i \leqslant n} \{\| x_1 - x_i \|\}, \qquad p = \infty,$$

and note that F is a gauge function on X^n. It is then easily verified that $\partial F(\bar{x}_1, \ldots, \bar{x}_n)$ is given by

$$\{(y_1, \ldots, y_n)/y_i \in S(X^*), \qquad i = 1, 2, \ldots, n \text{ and } y_i \text{ satisfying (i), (ii) and (iv)}\}.$$

The proof is completed by applying Theorem 2.1. ∎

ACKNOWLEDGMENTS

It is a pleasure to acknowledge indebtedness to Professor P. J. Laurent and Professor R. T. Rockafellar for stimulating discussions on the subject while the author was at l'I.R.M.A., Mathématiques appliquées-informatique, Grenoble, France, on a French Government Fellowship. Thanks are also due to Dr. F. Schurer of the Technological University Eindhoven, Netherlands, for his constant inspiration.

REFERENCES

1. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators," Part I, Interscience, New York, 1958.
2. C. B. DUNHAM, Simultaneous Chebyshev approximation of functions on an interval, *Proc. Amer. Math. Soc.* **18** (1967), 472–477.

3. R. B. Holmes, "A Course on Optimization and Best Approximation," Lecture Notes, No. 257, Springer–Verlag, Berlin, 1972.
4. J. L. Joly, Une famille de topologies et de convergences sur l'ensemble des fonctionnelles convexes, Thèse, Université de Grenoble, 1970.
5. J. L. Kelley, I. Namioka *et al.*, "Linear Topological Spaces," Van Nostrand, New York, 1963.
6. P. J. Laurent, "Approximation et Optimization," Hermann, Paris, 1972.
7. P. J. Laurent and Pham-Dinh-Tuan, Global approximation of a compact set by elements of a convex set, *Numer. Math.* **15** (1970), 137–150.
8. J. J. Moreau, Fonctionnelles convexes, Séminaire sur les Équations aux Dérivées Partielles, Collège de France (mimeographed lecture notes), Paris, 1966.
9. D. V. Pai, A characterization of smooth normed linear spaces, *J. Approximation Theory* **17** (1976), 315–320.
10. D. V. Pai, Proximal points of convex sets in normed linear spaces, *Yokohama Math. J.* **22** (1974), 53–78.
11. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, N. J., 1970.
12. I. Singer, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer–Verlag, Berlin, 1970.

Printed in Belgium